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# Estimating the uncertainty associated with Widmark's equation as commonly applied in forensic toxicology

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**Abstract**

No computation is performed more frequently by forensic toxicologists than that involving Widmark's equation. The equation is employed to estimate either the number of drinks consumed or the corresponding blood or breath alcohol concentration. Despite the wide use of Widmark's equation, rarely is an uncertainty estimate also provided. Estimates from Widmark's equation involve at least seven uncertain random variables. Uncertainty estimates are presented that rely on methods of general error propagation compared to a method developed by Widmark. Assuming reasonable variable and uncertainty estimates, the error propagation method yielded for $N = 10.4$ drinks, a combined uncertainty (standard deviation) of 1.3 drinks (CV = 12.3%). Similarly, estimating the blood alcohol concentration yielded for 0.120 g/100 ml, an uncertainty of 0.0255 g/100 ml (CV = 21.2%). Widmark's uncertainty method yielded 1.6 drinks (CV = 15.4%). The derivation of Widmark's uncertainty estimate is also presented, showing that he considered only $\rho$ and $\beta$ to be uncertain. Widmark estimates for the number of drinks should include a 2CV estimate of approximately 25% while the blood alcohol concentration estimate should include a 2CV estimate of approximately 42%. Including valid estimates of uncertainty should enhance the legal admissibility and confidence for Widmark estimations.


*Keywords:* Widmark's equation; Uncertainty; Error propagation

## 1. Introduction

The seminal work of E.M.P. Widmark, the early 20th century Swedish physician and alcohol researcher, continues to provide the foundation for our understanding of alcohol physiology and toxicology [1]. One of Widmark's most important contributions was his development of a basic equation used to estimate the amount of alcohol consumed [2]:

$$A = C_t\, pr \tag{1}$$

where $A$ is the amount of alcohol absorbed and distributed throughout the body at time of sampling (g), $C_t$ the blood alcohol concentration at time $t$ (g/kg), $p$ the body weight (kg), and $r$ is the reduced body mass (dimensionless).

While Widmark measured blood alcohol concentrations in units of mass/mass, modern analytical methods commonly report the units in mass/volume. Therefore, revisions to Widmark's equation include $C_t$ reported in grams per liter and $r$ (''apparent'' volume of distribution) reported in liters per kilogram. Widmark's equation continues to be widely used in forensic and research contexts including: (1) court proceedings where blood or breath alcohol evidence is introduced and estimates of alcohol consumption are requested, (2) experimental drinking labs as part of training courses for law enforcement and forensic personnel, (3) alcohol research studies, (4) development of alcohol consumption and blood alcohol concentration (BAC) nomograms, etc. Probably no computation in forensic toxicology is performed more frequently. Moreover, refinements to Widmark's equation continue to be of important research interest [3–9].

Despite the wide application of Widmark's equation in many contexts today, there seems to be little appreciation for its uncertainty. Estimates from the equation result from the mathematical computation of several measured variables whose uncertainties propagate. The only forensically appropriate way to present and interpret Widmark estimates is to include an assessment of their uncertainty. Failing to acknowledge uncertainty is probably most pronounced in the courts

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where juries are asked to consider and weigh the quantitative estimates. Often lacking an appreciation for quantitative uncertainty, juries tend to assign an unmerited amount of weight to the estimates. Forensic scientists, therefore, should be prepared to present a reliable estimate of uncertainty along with any Widmark estimates.

Widmark estimations also find wide application in determining the dose of alcohol to administer volunteers in experimental drinking labs for educational or research purposes. The educational understanding will be enhanced where uncertainty estimates accompany dose determinations. In addition, the ethical concerns associated with administering any drug for training or research purposes should motivate the consideration of uncertainty. Moreover, uncertainty estimates for alcohol dose and BAC target values may be required by Human Subject Research Institutional Review Boards as part of research approval or funding.

After acknowledging the need for uncertainty estimates, it is important to determine them as reliably as possible. Where previous work has proposed uncertainty estimates for Widmark's equation, they include only a limited number of variables from the model [10,11]. The purpose here will be to expand on previous work and present numerically reasonable estimates of the total uncertainty associated with Widmark's equation. These methods will also be compared to Widmark's own method for estimating uncertainty.

### 1.1. Deriving Widmark's equation

From Widmark's ability to measure alcohol in blood, he desired a mathematical expression allowing him to determine the total amount of alcohol absorbed and distributed within the body at the time of sampling. This relationship can be expressed as:

$$\frac{g_1}{V_d} = \frac{g_2}{V_2} \tag{2}$$

where $g_1$ is the grams of alcohol distributed in the entire body, $V_d$ the "apparent" volume of distribution in the body (L), $g_2$ the grams of alcohol in the blood, and $V_2$ is the volume of blood in liters.

In Eq. (2) both $V_d$ and $V_2$ have the same concentration of alcohol. $V_d$ represents the "apparent" volume of distribution for the entire body, that is, the total theoretical volume into which the absorbed alcohol ($g_1$) would need to distribute itself in order to yield the same concentration as that existing in the blood ($V_2$). The term "apparent" is associated with $V_d$ because this volume is not an anatomical reality. Since alcohol distributes into all body water, $V_d$ will be larger than that of the blood volume itself. Eq. (2) can then be re-expressed as:

$$\frac{g_1}{V_d} = \frac{g_2}{V_2} \Rightarrow g_1 = V_d \frac{g_2}{V_2} \tag{3}$$

Since people have different body weights, $V_d$ can be determined from the product of body weight and a reduced estimate, $r$, in L/kg. Widmark referred to $r$ as a "reduced body mass", representing the "apparent" volume of distribution per kg of body weight [2]. Eq. (3) can then be written in its common form:

$$g_1 = V_d \frac{g_2}{V_2} \Rightarrow A = WrC_0 \tag{4}$$

where $A$ is the amount of alcohol distributed in the body (g), $W$ the body weight (kg), $r$ the reduced volume of distribution (L/kg), and $C_0$ is the blood alcohol concentration (g/L).

While not advocated by Widmark, modern interest is in determining the specific number of drinks owing to the general awareness of different alcoholic beverages. The result is a more relevant form of Widmark's equation:

$$N = \frac{Wr[C_t + \beta t]}{dZ} \tag{5}$$

where $N$ is the number of drinks, $W$ the body weight (oz), $r$ the volume of distribution (L/kg), $C_t$ the blood alcohol concentration at time $t$ (kg/L), $\beta$ the alcohol elimination rate (kg/L/h), $t$ the time since drinking began (h), $d$ the density of alcohol (0.82 oz/fl oz), and $Z$ the fluid ounces of ethyl alcohol per drink.

Rearranging Eq. (5) to solve for BAC as a function of the number of drinks yields:

$$C_t = \frac{NdZ}{Wr} - \beta t \tag{6}$$

Eqs. (5) and (6) represent the most commonly used forms in forensic and research contexts today.

### 1.2. Determining the model's uncertainty

Eqs. (5) and (6) reveal seven uncertain random variables. When the model and error structure of the variables are known or reasonably assumed, a sound mathematical foundation exists for estimating uncertainty. This is illustrated by estimating the total uncertainty for the number of drinks computed in Eq. (5). Table 1 shows the variables assumed for this example along with their uncertainty estimates (standard deviations). The values in Table 1 are for illustration only and should not be assumed relevant for all cases. Each case would need to provide the best estimates available for their particular subject. While some of the uncertainties assumed in Table 1 are arbitrary, they seem forensically reasonable for illustrative purposes. The standard deviation for the blood alcohol concentration ($S_{C_t}$) was assumed to be that for breath alcohol measurement determined from a large number of duplicate results [13]. Breath alcohol uncertainty was employed, rather than blood, because it is most commonly employed in Widmark estimations. Employing blood alcohol results would require a smaller uncertainty estimate for $C_t$ [14]. Incorporating the values from Table 1, Eq. (5) yields an estimate of 10.4 drinks (drinks were 12 fluid ounce beers at 4% alcohol by volume). Since the variables in Widmark's equation are both additive and multiplicative, the variances cannot be combined simply by adding directly ($S_N = \sqrt{S_W^2 + S_r^2 + \cdots + S_Z^2}$) or relatively ($\mathrm{CV}_N = \sqrt{\mathrm{CV}_W^2 + \mathrm{CV}_r^2 + \cdots + \mathrm{CV}_Z^2}$). Instead, the appropriate method for estimating uncertainty is the delta method or the

Table 1
The assumed values and uncertainties for the variables introduced into Widmark's equation for estimating the number of drinks consumed

| Variable | Value | Uncertainty (S.D.)[a] | % CV[b] |
|---|---|---|---|
| Gender | Male | | |
| Weight | 180 lbs | 3.6 lbs | 2 |
| $r$ | 0.73 L/kg[c] | 0.067 L/kg[c] | 9.2 |
| $C_t$ | 0.0012 kg/L | 0.000043 kg/L[d] | 3.6 |
| $\beta$ | 0.000148 kg/L/h[e] | 0.000032 kg/L/h[e] | 22 |
| $t$ | 5 h | 0.1 h | 2 |
| $d$ | 0.82 oz/fl oz | 0.0082 oz/fl oz | 1 |
| $Z$ | 0.48 fl oz/drink[f] | 0.014 fl oz/drink | 3 |

[a] Estimated standard deviations for a single measurement.
[b] Coefficient of variation.
[c] Values reported in the study of $n = 24$ males [12].
[d] Assumed estimated from duplicate breath alcohol analyses [13].
[e] Values reported in the study of $n = 25$ males [12].
[f] The assumed drink was a 12 fluid ounce beer with 4% alcohol by volume.

general method of error propagation [15,16]:

$$S_N = \sqrt{\left[\frac{\partial N}{\partial W}\right]^2 S_W^2 + \left[\frac{\partial N}{\partial r}\right]^2 S_r^2 + \left[\frac{\partial N}{\partial C_t}\right]^2 S_{C_t}^2 + \left[\frac{\partial N}{\partial \beta}\right]^2 S_\beta^2 + \left[\frac{\partial N}{\partial t}\right]^2 S_t^2 + \left[\frac{\partial N}{\partial d}\right]^2 S_d^2 + \left[\frac{\partial N}{\partial Z}\right]^2 S_Z^2 + 2\frac{\partial N}{\partial r}\frac{\partial N}{\partial \beta}\mathrm{Cov}(r,\beta)} \quad (7)$$

where $S_N$ is the standard deviation of the number of drinks, $\mathrm{Cov}(r,\beta)$ is the covariance between $r$ and $\beta$.

By employing the first derivatives, Eq. (7) is a linear approximation that ignores higher order derivatives and their influence. This has no effect on first degree variables existing in the numerator, since higher order derivatives are zero. Variables existing in the denominator, however, have higher order derivatives that are ignored, and yet may have some (small) influence. Eq. (7) also includes one covariance term because of the correlation observed between $r$ and $\beta$ ($\hat{\rho} = -0.135$) [10]. All other variables are assumed independent ($\hat{\rho} = 0$). Employing the estimates from Table 1, Eq. (7) yields $S_N = 1.27$ drinks, corresponding to a coefficient of variation (CV) of 12.3%. The proportional contribution of each variable to the total uncertainty is also shown in Table 2.

The other common use of Widmark's equation is computing an estimate for the BAC (Eq. (6)) assuming the individual consumed a specified number of drinks. If we assume $N = 10.4$ drinks (12 fluid ounce beers with 4% alcohol by volume) with a standard deviation of 0.52 drinks (CV = 5%) along with the other values from Table 1, Eq. (6) yields, as expected, $C_t = 0.0012$ kg/L = 0.12 g/100 ml. The reduced uncertainty in $N$ (5%CV compared to 12.3%) is because $N$ would not be estimated from several other variables. Employing Eq. (7), the one standard deviation uncertainty estimate for $C_t$ is 0.000255 kg/L, corresponding to a CV of 21.2%. This relative uncertainty estimate is significantly larger compared to when estimating the number of drinks ($N$). Table 2 summarizes the contribution of each variable to the uncertainty for estimating either $N$ or $C_t$ where the contribution was based on the sum in Eq. (7) prior to subtracting the covariance term.

### 1.3. Widmark's estimate of uncertainty

Widmark, appreciating the importance of uncertainty, derived separate equations for men and women to estimate one standard deviation estimates in $A$ as follows [2]:

$$\text{for men}: \quad S_A = \sqrt{0.015625A^2 + 0.050176(A - 0.68C_t p)^2} \quad (8)$$

$$\text{for women}: \quad S_A = \sqrt{0.01A^2 + 0.021904(A - 0.55C_t p)^2} \quad (9)$$

Table 2
The proportion that each variable in Widmark's equation contributes to estimating the uncertainty in either the number of drinks or the blood alcohol concentration

| Estimation of $N$ ($N = 10.4$ drinks) | | | Estimation of $C_t$ ($C_t = 0.0012$ kg/L) | | |
|---|---|---|---|---|---|
| Variable | Value (CV) | Percent of total uncertainty[a] | Variable | Value (CV) | Percent of total uncertainty[a] |
| $W$ | 180 lbs (2%) | 2.3 | $W$ | 180 lbs (2%) | 2.1 |
| $r$ | 0.73 L/kg (9.2%) | 49.1 | $r$ | 0.73 L/kg (9.2%) | 44.0 |
| $C_t$ | 0.0012 kg/L (3.6%) | 2.9 | $N$ | 10.4 (5%) | 13.1 |
| $\beta$ | 0.000148 kg/L/h (21.6%) | 39.6 | $\beta$ | 0.000148 kg/L/h (21.6%) | 35.3 |
| $t$ | 5 h (2%) | 0.3 | $t$ | 5 h (2%) | 0.3 |
| $d$ | 0.82 oz/fl oz (1%) | 0.6 | $d$ | 0.82 oz/fl oz (1%) | 0.5 |
| $Z$ | 0.48 fl oz/drink (3%) | 5.2 | $Z$ | 0.48 fl oz/drink (3%) | 4.7 |
| Total | | 100 | | | 100 |

[a] These are percentages of the sum in Eq. (7) prior to subtracting the covariance term.

The derivation of these equations is given in Appendix A. While Widmark applied the correct approach in deriving his uncertainty equations from the general error propagation method [15,16], he included the uncertainty from only two variables: $r$ and $\beta$. Moreover, he neglected to include the covariance term in his uncertainty estimate, resulting in an over estimate of $S_A$ due to the negative correlation between $r$ and $\beta$ [10]. Substituting $N$ for $A$ in Eq. (8), yields a more relevant form for Widmark's uncertainty equation:

$$\text{for men}: \quad S_N = \sqrt{0.015625N^2 + 0.050176\left(N - \frac{WrC_t}{dZ}\right)^2} \tag{10}$$

Employing Eq. (10) for the example from Table 1 and using $r = 0.73$ L/kg rather than Widmark's dimensionless value of 0.68 for men yields: $S_N = 1.6$ drinks, corresponding to CV of 15.4%. Alha [11] also derived an uncertainty estimate based only on $r$ and $\beta$ having the same form as Widmark's:

$$S_N = \sqrt{0.02578A^2 + 0.07749(A - 1.02C_tW)^2} \tag{11}$$

The derivation for Alha's estimate in Eq. (11) is found in Appendix B. Unlike Widmark, Alha accounted for the covariance term between $r$ and $\beta$ in his model, resulting in a more accurate estimate. Alha's estimate of one standard deviation for the example would be: $S_N = 0.8$ drinks, corresponding to a CV of 8.0%. Table 3 compares the three estimates of uncertainty applied to the example found in Table 1.

Appendices A and B show the constants in both Widmark and Alha's equations (Eqs. (10) and (11)) are derived from data observed by each investigator. Rather than employing their coefficient estimates, those assumed in Table 1 can be incorporated. The resulting uncertainty estimates are also shown in Table 3 along with those obtained when using the published coefficients of Widmark and Alha. The close correspondence between Widmark's and Alha's estimates (1.29 and 1.20, respectively) using the same data from Table 1 can be attributed to the small assumed correlation between $r$ and $\beta$ ($\hat{\rho} = -0.135$). As this correlation becomes more negative, Alha's estimates become significantly smaller.

Table 3
One standard deviation uncertainty estimates for the number of drinks ($N = 10.4$ drinks) determined from Widmark's equation based on the three methods of estimation

| Method | One standard deviation | Coefficient of variation (%) |
|---|---|---|
| Error Propagation | 1.27 drinks | 12.3 |
| Widmark[a] | 1.6 drinks | 15.4 |
| Alha[b] | 0.8 drinks | 8.0 |
| Widmark[c] | 1.29 drinks | 12.4 |
| Alha[c] | 1.20 drinks | 12.0 |

[a] Using the coefficients determined by Widmark in Eq. (8).
[b] Using the coefficients determined by Alha in Eq. (11).
[c] Using the values assumed in Table 1 with the general models for Widmark and Alha developed in Appendices A and B.

## 2. Discussion

The analyses presented here show significant uncertainty is associated with estimates from Widmark's equation. While both Widmark and Alha derived their estimates appropriately from the general method of error propagation, they both considered only two variables as being uncertain—$r$ and $\beta$. Although these two variables contribute 80% or more to the total uncertainty (Table 2), their methods fail to include the uncertainty due to five other components of the equation. The general method of error propagation appropriately includes all variables and, for the example, yields $S_N = 1.27$ drinks (CV = 12.3%) while Alha's method yields $S_N = 0.8$ drinks (CV = 8.0%). Alha's method is significantly smaller because he used a correlation estimate of $\hat{\rho} = -0.58$ [11]. By failing to include the covariance term in Eq. (7), Widmark's estimate ($S_N = 1.6$ drinks, CV = 15.4%), on the other hand, is too large. Using the same assumed values from Table 1 to determine the coefficients of both Widmark's and Alha's uncertainty equations yields nearly identical estimates to that of the general error propagation method (Table 3). This corresponds with Table 2 showing that the additional five variables contribute very little (about 10%) to the total uncertainty. In addition, the close correspondence in Table 3 suggests that the use of $\hat{\rho} = -0.135$ in both the error propagation and Alha's method, while not in Widmark's, makes little difference. However, if the correlation between $r$ and $\beta$ employed by Alha ($-0.58$) is more reliable than that assumed in Table 1 ($-0.135$), its use would further reduce the general error propagation estimate as well. While a biological explanation for the correlation between $r$ and $\beta$ is not well established, a great deal of research has shown that women, with smaller $r$ values, generally show larger values for $\beta$ compared to men. While neither Widmark nor Alha provided uncertainty equations for estimating $C_t$, these estimates are easily determined from the general method of error propagation in Eq. (7). Finally, it was unnecessary for Widmark to obtain two uncertainty equation estimates (one for men and one for women). Using the general method of Eq. (7) provides one equation for including any estimates for the variables regardless of gender.

While the example presented here employed the standard Widmark model, others have proffered revised models involving total body water (TBW) [9]. The value of $V_d$ can be estimated either by Widmark's value ($Wr$) or by TBW according to:

$$V_d = Wr = \frac{\text{TBW}}{0.8} \tag{12}$$

where $V_d$ is the total "apparent" volume of distribution (L), TBW the total body water (L), and 0.8 is the water fraction of blood (L/L).

The method of TBW can also be employed in Widmark estimations along with appropriate uncertainty estimates using Eq. (7), as long as valid uncertainty estimates were available for TBW and the water fraction of blood. Indeed, this may yield different uncertainty estimates for $N$ and $C_t$.

Employing Widmark's equation includes several assumptions that should be acknowledged. One assumption is employing a breath alcohol measurement directly as blood alcohol by assuming $K_{BAC/BrAC} = 2100$. Clearly, there is uncertainty in this assumption that can also be incorporated into Eq. (7), along with a more reliable estimate for $K_{BAC/BrAC}$. When the values of $K_{BAC/BrAC} = 2407$ and $S_K = 213$ [17] were incorporated as another term in Eq. (7), the result was $S_N = 1.43$ drinks (CV = 13.8%)—a slight increase compared to the model without $K_{BAC/BrAC}$ (Table 3). In addition, the uncertainty in $K_{BAC/BrAC}$ contributed 18.6% to the total uncertainty of $N$ which, while significant, still were less than the contributions from $r$ and $\beta$. While Widmark did not consider $K_{BAC/BrAC}$, working with blood alcohol only, he did not even consider analytical uncertainty in his Eqs. (8) and (9). However, appropriately including blood alcohol measurements and their uncertainty would clearly reduce the uncertainty in both $N$ or $C_t$ compared to breath alcohol results. Another important assumption is that the subject is in the post-absorptive state, which Widmark believed should not be assumed before at least 1.5 h after the last drink [2]. The uncertainty estimates determined from Eq. (7) also depend significantly on the standard deviations for each variable – often difficult to estimate reliably. The values assumed in Table 1 could certainly be refined through experimental or literature sources, yielding improved uncertainty estimates for $N$ and $C_t$. Employing literature sources of variable uncertainties should represent between-subject estimates since the goal is to make inference regarding an individual assumed randomly selected from the general population. Pooling literature estimates in the form of meta-analysis is another approach for improving variable and uncertainty estimates.

Several other factors influencing the uncertainty in Widmark estimates were not considered here either. These include: (1) effects of food quantity and type, (2) the limitations of $r$ as a measure of true volume of distribution, (3) first-pass metabolism, (4) gastric emptying rates, (5) bioavailability, (6) position on the alcohol concentration time curve, etc. Instead, the example and associated computations presented rely solely on the usual Widmark assumptions of, for example: (1) bolus dosing on an empty stomach, (2) complete alcohol absorption and distribution and (3) at least 1.5 h post drinking sampling time. Indeed, the forensic scientist should acknowledge these additional uncertainty elements. However, they do not represent measurable variables within Widmark's equation and therefore have no quantitative uncertainties assigned to them. Instead, the computations presented here represent the most common applications of Widmark's equation today.

The analyses presented here also reveal that, despite the use of the same independent variables, the relative uncertainty estimates for $N$ and $C_t$ are not the same. While using reasonable variable estimates in the error propagation method, the CV for $C_t$ (21.2%) remained larger than for $N$ (12.3%). While Table 2 shows each variable contributing the same relative amount to the uncertainty in $N$ and $C_t$, $N$ contributes a larger fraction to the uncertainty in $C_t$ than $C_t$ contributes to the uncertainty in $N$. Because each variable and uncertainty interacts in a complex mathematical way in Eq. (7), it is difficult to predict which combination might alter the relative uncertainty estimates for $N$ and $C_t$. A variable having a large uncertainty may not contribute significantly due to the magnitude of the partial derivative coefficient. These coefficients measure sensitivity and influence the estimate as well. Table 1, for example, shows that while $\beta$ had more than twice the relative uncertainty compared to $r$ (22% compared to 9.2%), the contribution of $\beta$ to the uncertainty of either $N$ or $C_t$ was less than that of $r$. Forensic scientists should simply be aware that differences in uncertainty exist when computing either $N$ or $C_t$. For those interested, the mathematical details may be studied further employing computer simulations. Instead, the focus should be on employing the appropriate methods (equations) for obtaining the best possible estimates for each variable and its uncertainty when solving for either $N$ or $C_t$.

While forensic toxicologists should be prepared to comprehend and compute the uncertainty estimates presented here, it is not expected that they would explain these details in the courtroom context. The technical details would not be relevant to the court's decisions. The forensic toxicologist, however, should appreciate and understand the importance of uncertainty in numerical calculations and be able to obtain these estimates employing a sound numerical approach.

## 3. Conclusion

Employing seven or eight uncertain variables in Widmark's equation clearly results in estimates having significant uncertainty that should be acknowledged when employed by forensic toxicologists. Assuming reasonable estimates for the variables and their uncertainties as presented here, a 2CV uncertainty interval of ±25% should be applied when reporting estimates of the number of drinks. When reporting an estimated BAC, a 2CV uncertainty interval should be approximately ±42%. Improved estimates would result if each variable were to be measured and reliable uncertainty estimates determined. However, based on what has been presented here, forensic toxicologists should be able to incorporate their own experimental data, or literature estimates, and arrive at reasonable uncertainty estimates for Widmark's equation. Moreover, these methods can be easily applied in widely available spreadsheet and computer simulation programs. Developing and reporting appropriate uncertainty estimates when applying Widmark's equation should improve confidence and acceptability in both research and legal contexts.

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## Appendix A. Deriving Widmark's uncertainty estimate

Widmark expressed his general equation as:

$$A = pr(C_t + \beta t) \quad \text{(A.1)}$$

where $A$ is the amount of alcohol consumed (g), $p$ the body weight (kg), $r$ the volume of distribution (L/kg), $C_t$ the blood alcohol concentration at time $t$ (g/L), $\beta$ the blood alcohol elimination rate (g/L/h), $t$ is the time since consumption began (h).

He also presented an equation for estimating one standard deviation ($S_A$) based on his general model in Eq. (A.1):

$$S_A = \sqrt{K_1 A^2 + K_2 (A - rC_t p)^2} \tag{A.2}$$

where $A$ is the amount of alcohol consumed as estimated from Eq. (A.1) (g), $K_1$ a dimensionless constant, and $K_2$ a dimensionless constant.

Widmark developed two forms for Eq. (A.2), one for men and one for women, differing in the values of $K_1$, $K_2$ and $r$. He apparently determined that the uncertainty in his model (Eq. (A.1)) was determined largely by $r$ and $\beta$, also considered to be independent. He further recognized that the general error propagation equation, derived from the Taylor series [16,17], would apply. Thus, employing the Taylor series, he expressed his model as:

$$\begin{aligned} A &= f(r, \beta) \\ &\approx f(r_0, \beta_0) + (r - r_0)\left.\frac{\partial A}{\partial r}\right|_{r_0,\beta_0} + (\beta - \beta_0)\left.\frac{\partial A}{\partial \beta}\right|_{r_0,\beta_0} \end{aligned} \tag{A.3}$$

Since both $r$ and $\beta$ are linear in $A$, all higher order derivatives are zero and need not be included. Widmark then took the variance of both sides to find:

$$\begin{aligned} V(A) &= V[f(r,\beta)] \approx V[f(r_0,\beta_0)] + V\left[(r - r_0)\left.\frac{\partial A}{\partial r}\right|_{r_0,\beta_0}\right] + V\left[(\beta - \beta_0)\left.\frac{\partial A}{\partial \beta}\right|_{r_0,\beta_0}\right] \\ V(A) &\approx \left[\left.\frac{\partial A}{\partial r}\right|_{r_0,\beta_0}\right]^2 V(r) + \left[\left.\frac{\partial A}{\partial \beta}\right|_{r_0,\beta_0}\right]^2 V(\beta) \end{aligned} \tag{A.4}$$

Eq. (A.4) represents the general error propagation equation considering two independent variables only. From Eq. (A.1), he found the partial derivatives for Eq. (A.4), and recognizing that $V(Y) = (Y\mathrm{CV}_Y)^2$ (CV = coefficient of variation), he then re-expressed Eq. (A.4) as follows:

$$\begin{aligned} V(A) &\approx [WC_t + W\beta t]^2 V(r) + [Wrt]^2 V(\beta) \\ &\approx \mathrm{CV}_r^2 W^2 r^2 [C_t + \beta t]^2 + \mathrm{CV}_\beta^2 [Wr\beta t]^2 \end{aligned} \tag{A.5}$$

From Eq. (A.5) he recognized the value of $A$ in both terms and arrived at his final form for the one standard deviation uncertainty estimate:

$$\begin{aligned} V(A) &\approx \mathrm{CV}_r^2 A^2 + \mathrm{CV}_\beta^2 [A - WrC_t]^2 \Rightarrow S_A \\ &\approx \sqrt{\mathrm{CV}_r^2 A^2 + \mathrm{CV}_\beta^2 [A - WrC_t]^2} \end{aligned} \tag{A.6}$$

From Eq. (A.6) we see that the dimensionless constants in Eq. (A.2) are: $K_1 = \mathrm{CV}_r^2$ and $K_2 = \mathrm{CV}_\beta^2$. These values differed for the men and women in Widmark's study as they would for any group of experimental subjects. Based on the experimental results from the subjects studied by Widmark, he derived his uncertainty equation for men and women of:

$$\text{for men:} \quad S_A = \sqrt{0.015625A^2 + 0.050176(A - 0.68C_t p)^2} \tag{A.7}$$

$$\text{for women:} \quad S_A = \sqrt{0.01A^2 + 0.021904(A - 0.55C_t p)^2} \tag{A.8}$$

## Appendix B. Deriving Alha's uncertainty estimate

Alha also began with Widmark's general equation and, like Widmark, assumed only $r$ and $\beta$ to be uncertain. He then derived a two standard deviation estimate for uncertainty having the general form:

$$2S_A = \sqrt{K_1 A^2 + K_2 (A - K_3 C_t p)^2} \tag{B.1}$$

where $A$ is the amount of alcohol consumed as estimated from Eq. (A.1) (g), $K_1$ a dimensionless constant, $K_2$ a dimensionless constant, and $K_3$ a dimensionless constant.

Alha also began with Taylor's approximation but, unlike Widmark, included the covariance term, apparently realizing that $r$ and $\beta$ were correlated:

$$\begin{aligned} V(A) &\approx \left[\left.\frac{\partial A}{\partial r}\right|_{r_0,\beta_0}\right]^2 V(r) + \left[\left.\frac{\partial A}{\partial \beta}\right|_{r_0,\beta_0}\right]^2 V(\beta) \\ &\quad + 2\left.\frac{\partial A}{\partial r}\right|_{r_0,\beta_0}\left.\frac{\partial A}{\partial \beta}\right|_{r_0,\beta_0} \mathrm{Cov}(r,\beta) \end{aligned} \tag{B.2}$$

where $\mathrm{Cov}(r,\beta) = S_r S_\beta \hat{\rho}$ and $\hat{\rho}$ is the correlation between $r$ and $\beta$.

The first two terms of Eq. (B.2) simplify to the same expression as shown in Eq. (A.5). Adding the covariance term yields:

$$\begin{aligned} V(A) &\approx \mathrm{CV}_r^2 W^2 r^2 [C_t + \beta t]^2 + \mathrm{CV}_\beta^2 [Wr\beta t]^2 \\ &\quad + 2\hat{\rho} Wr(C_t + \beta t)(Wr\beta t)\mathrm{CV}_r \mathrm{CV}_\beta \end{aligned} \tag{B.3}$$

Eq. (B.3) further simplifies to:

$$\begin{aligned} V(A) &\approx \mathrm{CV}_r^2 A^2 + 2\hat{\rho} Wr(C_t + \beta t)(Wr\beta t)\mathrm{CV}_r \mathrm{CV}_\beta \\ &\quad + \mathrm{CV}_\beta^2 [A - WrC_t]^2 \approx \mathrm{CV}_r^2 A^2 + 2\hat{\rho}\mathrm{CV}_r \mathrm{CV}_\beta A[A - WrC_t] \\ &\quad + \mathrm{CV}_\beta^2 [A - WrC_t]^2 \end{aligned} \tag{B.4}$$

Eq. (B.4) has the general form:

$$ax^2 + 2bx[x - y] + c[x - y]^2 \tag{B.5}$$

where $a = \mathrm{CV}_r^2$, $b = \hat{\rho}\mathrm{CV}_r\mathrm{CV}_\beta$, $y = WrC_t$, and $c = \mathrm{CV}_\beta^2$.

Eq. (B.5) can be re-expressed as:

$$ax^2 + 2bx[x - y] + c[x - y]^2 = \left(a - \frac{b^2}{c}\right)x^2 + \frac{(b+c)^2}{c}\left[x - \frac{c}{b+c}y\right]^2 \tag{B.6}$$

Putting the values from Eq. (B.4) into Eq. (B.6) and simplifying yields the variance of $A$:

$$V(A) = [\mathrm{CV}_r^2(1 - \hat{\rho}^2)]A^2 + (\hat{\rho}\mathrm{CV}_r + \mathrm{CV}_\beta)^2\left[A - \left(\frac{\mathrm{CV}_\beta}{\hat{\rho}\mathrm{CV}_r + \mathrm{CV}_\beta}\right)WrC_t\right]^2 \tag{B.7}$$

Notice that if $\hat{\rho} = 0$ (implying independence between $r$ and $\beta$ as assumed by Widmark), Eq. (B.7) reduces to that of Widmark's form of Eq. (A.6). Using the results from his experimental subjects, multiplying Eq. (B.7) by four to obtain a $2S_A$ estimate and incorporating the value of $r$ into the coefficient of Eq. (B.7), Alha arrived at his uncertainty estimate of:

$$S_N = \sqrt{0.02578A^2 + 0.07749(A - 1.02C_tW)^2} \tag{B.8}$$

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